Mathematical attitude and the liberal arts

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Abstract

One of the reasons for studying mathematics is to acquire the ability to think freely.

In mathematics, one is free to formulate hypotheses as long as they are consistent (called axioms in mathematics). For example, in non-Euclidean geometry, the sum of the interior angles of a triangle can be greater than or less than 180 degrees. At first glance, this may seem counterintuitive, but it is equivalent to Euclidean geometry in terms of consistency. On the contrary, non-Euclidean geometry is an indispensable theory for the theory of relativity.

Hypothesis is not a concept used only in mathematics and science. Many hypotheses are used in our thinking, both explicitly and implicitly. Hypotheses enable us to think, but sometimes they limit our thinking.

In the former sense, a person's hypothesis is his/her worldview. In response to this positive aspect of hypotheses, we call the attitude of confronting the world with the idea that "everything is a hypothesis" the "mathematical attitude" in this study. The reason for this is that in mathematics, there is freedom in setting up hypotheses except for consistency, and all proofs of mathematical theorems start from axioms.

The latter can be freed by becoming aware of the implicit hypotheses within oneself and relativizing or abstracting them. This awareness is the essence of thinking freely. Therefore, it is an important part of the liberal arts.

In this paper, we will discuss the relationship between "mathematical attitude" and liberal arts, and report on the importance of mathematics education.

The development of technology is changing society. The great social changes that have taken place in the past cannot be separated from technological innovations. In addition, the development of technology and the development of theory, that is, changes in hypotheses, are closely related. Therefore, "mathematical attitude" is especially important in engineering education at KOSEN.

Keywords: *hypothesis, axiom, mathematical attitude, liberal arts, pedagogy, National Institute of Technology*

Introduction

This paper proposes the concept of "mathematical attitude" in response to the change in the way mathematics is understood in connection with the discovery of non-Euclidean geometry. It also discusses the effects of mathematical attitudes and the relationship between mathematical attitudes and liberal arts education.

2.1 Euclidean geometry

Euclidean geometry consists of five postulates:

(i) A straight line can be drawn between any different two points.
(ii) A terminated line can be extended indefinitely.
(iii) A circle can be drawn with any given point as a center and any given radius.
(iv) All right angles are equal to one another.
(v) (Parallel postulate) A straight line can be uniquely drawn through a given point not on a given line and does not meet the given line.

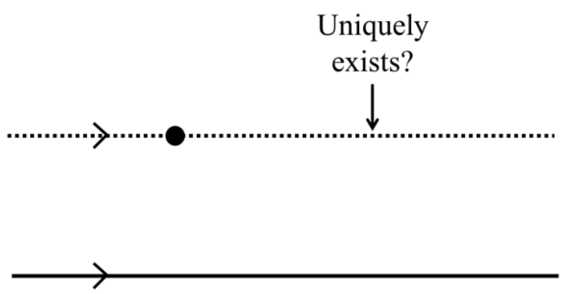

Figure 1 Parallel postulate

2.2. Relativisation of Euclidean geometry

Because the parallel postulate is non-trivial compared to other postulates, many attempts have been made to prove the parallel postulate from the other four postulates for over 2000 years. In the 18th century, G. G. Saccheri (1667-1733), an Italian Jesuit priest and mathematician, regarded the parallel postulate as a truth and argued that it cannot be proved directly. Instead, he tried to prove the parallel postulate indirectly by the method of contradiction. That is, he tried to derive a contradiction by assuming the negation of the parallel postulate. Although he derived many counterintuitive propositions from the negation of the parallel postulate, in the end he was unable to derive a contradiction.

In order to derive a proposition from the negation of the parallel postulate, we need a mind that follows logic thoroughly, not experience or intuition. In that sense, it can be said that Saccheri had a spirit that valued logic. In fact, he was also a logician. On the other hand, he could not distance himself from his belief that the parallel postulate is absolutely correct. The “contradiction” that he thought that he had been able to derive was not actually a contradiction. They are just propositions derived from the negation of the parallel postulate. It only seemed contradictory because he implicitly assumed a variant of the parallel postulate.

On the other hand, N. I. Lobachevsky (1792-1856), J. Bolyai (1802-1860), and J. C. F. Gauss (1777-1855) interpreted the counterintuitive propositions derived by Saccheri as *new theorems*. Moreover, it was a *new geometry* incompatible with Euclidean geometry. This new geometry has been called non-Euclidean geometry. These two geometries differ in intuitive acceptability but are equivalent in a logical sense. That is, if one is consistent, so is the other.

3.1. Discussion

What is the difference between Saccheri and mathematicians before him and Lobachevsky, et al.? It can be said that the difference in how to perceive the postulates. Saccheri considered them absolutely true, while Lobachevsky and others could think of them as one of many possible hypotheses. This difference gave birth to non-Euclidean geometry. Therefore, we refer to postulates and axioms here as hypotheses to emphasize their sense of being independent of experience and intuition.

This liberal attitude towards postulates is an important spirit of modern mathematics. It also has a different aspect from the scientific spirit. This is because mathematics and science have different objects. While science focuses on reality, the target of mathematics is often not only reality but rather mathematics itself. Mathematics can thus be constructed independently of reality. All that is required of mathematics is that the set of hypotheses (axiom system, a set of postulates) be consistent. As H. Poincare (1854-1912) pointed out in Poincare (1902), hypotheses in mathematics are merely *conventions*.

3.2. A new notion “Mathematical attitude”

Considering the free attitude toward hypotheses in mathematics, as mentioned above, we define the *mathematical attitude* as *attitudes facing the world based on the recognition that everything is based on hypotheses*.

All theorems in mathematics are propositions that have been proven. All the proofs can be traced back to the axioms if we trace them according to logic. Therefore, if the axioms change, the theorem that can be proved also changes completely. By the research in mathematical logic, like the axiomatic set theory ZFC, there are axiomatic systems that serve as "standards", but any axiom system is valid as mathematics if it is consistent. For example, while Euclidean geometry is the basis of our intuition about space and naive physics, non-Euclidean geometry is also a necessary theory for the theory of relativity and therefore an important set of axioms underpinning modern technology.

From this, we can extend the concept of axioms in mathematics and grasp the meaning of "hypothesis" broadly. Concepts that form the basis of our daily thinking, behaviors, and habits can also be called hypotheses. Some of these are born with us, while others are made by education, experience and upbringing, that is, by society. Rather, it can be said that the latter accounts for the majority. Moreover, some of these are conscious and some are unconscious.

3.3. What mathematical attitude tells us

Of course, we need to have many hypotheses in order to lead a daily life. In addition, some hypotheses expand the possibilities of our thinking. In this sense, the hypothesis gives us the possibility to act and think *freely*. On the other hand, there are many hypotheses which *limit* our thinking. For example, fixed beliefs, a way of thinking that clings to past successful experiences, and a way of thinking that perceives the world as a dichotomy between good and evil. In this case, we are often not consciously accustomed to such hypotheses.

In order to discover such our own hidden hypotheses, we pursue studies and interact with others who have different hypotheses. As a major premise to enable such growth, it is necessary to acquire a mathematical attitude. By adopting a mathematical attitude, we can show the following:

1. One can learn a new way of thinking.
2. More free thinking becomes possible.
3. Make it possible to communicate with a wider range of others.
4. One can acquire the spirit of equality.
5. One will be able to find happiness in everyday life.

About 1. We cannot learn other ways of thinking as long as we believe that we are unconditionally right. Here, the mathematical attitude gives us the recognition that we are not necessarily right, and that the other person is not necessarily wrong. This is because both our thoughts and those of others are based on hypotheses, and hypotheses can always be disproved. This kind of recognition enables us to have the attitude of trying to acquire the way of thinking that we do not yet know.

About 2. In order to think freely, first of all, it is necessary to become aware of (unconscious) prejudices, beliefs and values that limit one's thinking, and secondly, through the 1 above, it is necessary to relativize them by acquiring diverse ways of thinking. Mathematical attitudes also function for the former. This is because the mathematical attitude, by relativizing an idea, makes us think about what the basis for that idea is. As a result, the underlying preconceptions and values may be revealed.

About 3. If we have fixed ideas or strong beliefs, we will have difficulty communicating with others. Of course, it is important to have our own ideas and worldviews. On the other hand, by always securing the possibility of relativization, it becomes possible to improve and broaden one's thoughts through communication with others. Therefore, it is possible to communicate with various people by recognizing that there are various ways of thinking, that is, there are various hypotheses, through the mathematical attitudes. The mathematical attitude means the attitude to temporarily accept the ideas of others as hypotheses.

About 4. From the mathematical attitude, everything begins with a hypothesis. Therefore, there is no proposition that holds unconditionally. Also, once a proposition is proved in a system of axioms, it is just a theorem. It has nothing to do with who proved it or what kind of person proved it. In that sense, people are equal. In particular, one might say that they are equal under mathematics.

About 5. Our daily life is supported by the establishment of countless assumptions. In normal times, they are difficult to recognize. On the other hand, in an emergency, for the first time, we may notice the assumptions that support us. For example, the covid-19 turmoil has taught us the value of being able to meet and communicate with people and go on trips. But can we appreciate the value of everyday life without experiencing the extraordinary? Not necessarily. By hypothesizing all things in our daily lives through the mathematical attitudes, we can appreciate the fact that our daily lives exist, and we can be grateful for that. We can recognize the extraordinary of the ordinary through the mathematical attitudes. This is an important foundation for feeling gratitude and happiness in everyday life.

3.4. Mathematical attitude and liberal arts

From 1 and 2 above, we see that the mathematical attitude is relevant to liberal arts and its education. Liberal arts are a technique for becoming free, and for that purpose it is necessary to become aware of our own limitations and relativize them. In this sense, Horihata (2021) argued that the foundation of the liberal arts is the ability to abstract.

In order to abstract an object, it is necessary to first recognize the object. Next, it is necessary to have a viewpoint to abstract the object. It can be said that the liberal arts provide a perspective of abstraction backed by human history. This “viewpoint” corresponds to the “hypothesis” described in this paper.

Even with the same target, having different hypotheses will result in different information and value that can be found from the target. The result of the abstraction will also be different. In these days when countries are globalizing and values are diversifying, it can be said that being able to come and go between various viewpoints and worldviews is an important skill. Mathematical attitude underpins that ability.

By abstracting objects, we can sometimes find common structures among them. This enables communication and discussion between various people. Abstraction means discarding concepts that do not belong to the viewpoint. Therefore, for example, by noticing concepts that limit one's thinking and abstracting them away, one can think more freely.

Also, having an abstract interest means expanding the object of interest. Therefore, by making the interest more abstract, the amount of information that can be picked up increases. But it is not easy to have an abstract interest. We need to study so that we can draw abstract dreams and interests. These are done through language. In other words, the ability to use words freely in the language space is important. As I mentioned earlier, the characteristic of mathematics is that its object is also mathematics, that is, mathematics is purely a study in language space. Furthermore, mathematics can be said to be an academic discipline that explores the limits of freedom under rules, such as how freely theorems can be derived from fixed axioms and hypotheses in the language space. Therefore, in order to be free, it is very effective to explore mathematics and acquire a mathematical attitude.

4. Conclusions and future tasks

I focused on the fundamental structure of mathematics as an academic discipline, rather than simply as a discipline of logical thinking. In other words, mathematics is based on a system of axioms, and furthermore, we focused on the freedom of mathematics, in which the axioms can be chosen freely as long as they are consistent. Based on these aspects of mathematics, I defined a mathematical attitude and described the things that can be derived from it. In particular, I stated that we can free our thinking through mathematical attitudes, and that we can understand what happiness is by recognizing the non-obviousness of everyday life.

In this paper, I have discussed the relativization aspect and freedom of the mathematical attitudes. In the future, I would like to consider the mathematical attitudes and creativity, and mathematical attitudes and diversity.

Regarding the former, for example, by hypothesizing and abstracting some of the conditions of an existing structure, it is possible to relate it to other structures or create a new structure.

The latter is related to the self-referential nature of mathematics as a discipline. For sciences other than mathematics and other disciplines, the object is "outside" the discipline. On the other hand, the object of mathematics is also mathematics. Mathematics is purely a study of language. Therefore, a theorem proved by a system of axioms has its own meaning and value. In science, on the other hand, the conclusions obtained are always applied to the natural world or human society, and the meaning and value are determined by whether or not it is appropriate. Therefore, the diversity of consequences is one of the important features of mathematics.

I also would like to propose the possibility of reforming the liberal arts into a broader, more modern version through the notion of the mathematical attitudes.

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References

Y. Horihata (2021). Mathematical attitude (2). *Yonago KOSEN research report*, 56, 11-15

H. Poincare (1902). *Science and hypothesis*. Paris: Ernest Flammarion.